Fatal Problems Regarding The Momentum Operator In Position Space

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This text discusses a crucial problem, invalidating quantum mechanics. An unresolvable paradox, known from the onset of quantum mechanics[1–5], is discussed as an illustration, stemming from the said crucial problem. This is a sad state of affairs in science, kept carefully under wraps, and someone should finally come out and say that the Emperor has no clothes.

It is shown here that quantum mechanics fails at its very beginning, purely formally; that is, its very postulates fail. Quantum mechanics also fails physically even in its founding paper, ref.[6] The various problems in classical physics, purportedly calling for the emergence of quantum mechanics in the works of such figures as Rayleigh[7], Jeans[8] and Wien[9], to name a few, only show their own misinterpretations, rather than explicating problems in classical physics itself. The physical aspect of the failures in understanding classical physics, which led to the unnecessary emerging of quantum mechanics, will be discussed elsewhere. Finding fatal problems in the formal structure of quantum mechanics is sufficient for its removal from science. Returning to the classical roots of physics is the only way to set it back on the road of reason and consistency, as mandated by the scientific method.

To avoid confusion, we will be working here only in **position space**. Similar arguments apply if we choose to work in momentum space, which can attained by using Fourier transformations.

Even if we accept all that follows:

- the wave-function $\psi(x)$ contains all the information about position, momentum or perhaps energy of the system,
- the operators $\widehat{O}$ represent observables, such as position, momentum or perhaps energy,
- accept, as postulated, that these operators must be Hermitean; that is, that they must equal their conjugate-transpose, $\widehat{O} = \overline{\widehat{O}}^T$,

we get into irresolvable difficulties regarding the remainder of the postulates forming the foundation of quantum mechanics. This is prior to even discussing the postulate about the time-development of the system according to Schrodinger's equation.

Take, for instance, the eigenfunction equation for the momentum in position space of a free particle (potential $V(x) = 0$):

$$\widehat{p}\psi_p(x) = p\psi_p(x), \qquad (1)$$

in which quantum mechanics specifically postulates that the momentum operator in position space be expressed as[13] $\widehat{p} = -i\hbar\frac{\partial}{\partial x}$, replacing eq.(1) with the following equation:

$$-i\hbar\frac{\partial}{\partial x}\psi_p(x) = p\psi_p(x), \qquad (2)$$

where p is the eigenvalue and $\psi_p(x)$ is the eigenfunction of the momentum operator $\widehat{p}$ in position space. This is part of the backbone of quantum mechanics. However, this is exactly where the catastrophic failure of quantum mechanics occurs. The catastrophe is a result of what comes out as the eigenfunction of eq.(2); that is,

$$\psi_p(x) = Ce^{+i\frac{p}{\hbar}x}, \qquad (3)$$

as the function satisfying eq.(2), where C is a constant.

Unfortunately, however, if we treat with Born, eigenfunction (3) as <u>probability amplitude</u>, then the <u>probability density</u> will be

$$\psi_p^*(x)\psi_p(x) = Ce^{-i\frac{p}{\hbar}x}Ce^{+i\frac{p}{\hbar}x} = C^2, \qquad (4)$$

from where <u>probability</u> itself to find the particle in the region between x and $x + dx$ will be

$$\psi_p^*(x)\psi_p(x)dx = Ce^{-i\frac{p}{\hbar}x}Ce^{+i\frac{p}{\hbar}x}dx = C^2dx, \qquad (5)$$

which means that the probability to find the particle anywhere in space is uniform. This is clearly non-physical—the particle must be somewhere in space and be absent everywhere else. This well-known result, albeit usually understated, is a clear failure of the eigenfunction postulate of quantum mechanics to produce a physically valid state-function.

The importance of the above problem should not be minimized, as is usually done, because as a result of the above failure, most unfortunately, the discussed wave-function $\psi_p(x) = Ce^{+i\frac{p}{\hbar}x}$, eq.(3), further fails in that it cannot be normalized. The attempt to normalize it leads to an improper integral, which does not converge:

$$\int_{-\infty}^{+\infty} \psi_p^*(x)\psi_p(x)dx = \int_{-\infty}^{+\infty} C^2 dx = C^2 \int_{-\infty}^{+\infty} dx = \infty. \quad (6)$$

This in bra-ket notation has the following form[14]

$$\langle \psi_p | \psi_p \rangle = \infty \quad (7)$$

Now, we can stop right here because this is all that quantum mechanics derives for this problem—a non-physical state function.

The above presentation of the postulate and the outcome from it is well-known in the standard literature. The obvious problem it poses, however, is glossed over. The emphasis in the standard literature is shifted onto handling the entity $\psi_p(x) = Ce^{+i\frac{p}{\hbar}x}$, which itself fails as a state-function, and, therefore, is non-physical, to play the role of something physically sensible in such a way that further entities, which have it as building block, can somehow acquire physical meaning, nevertheless.

It should be emphatically stated, however, that the minute the above revelation that the eigenfunction equation, eq.(2), has non-physical solutions, that should be enough to no less than preclude any further consideration of quantum mechanics. The necessity to ignore quantum mechanics after such a stumbling block has nothing to do with disregarding innovation and creativity of thought, the way abandoning the flat Earth theory or geocentrism is expected in science without a second thought.

One should very clearly realize that the fact that the eigenfunction ψ_p (ψ_x as well) is not contained in Hilbert space and is not square-integrable in any Hilbert space, rigged or not, leading to the Lebesgue integral value of ∞, is a fatal problem of quantum mechanics itself and should not be manipulated to appear that it is not a problem. This non-physical outcome, arrived at when applying mathematical procedures in full concordance with the rules of quantum mechanics, is enough to invalidate it as a scientific theory.

Note, therefore, that once it is established, as we did above, that ψ_p (ψ_x as well) is not contained in Hilbert space because it is not normalizable (ψ_p, as well as ψ_x, are not normalizable in rigged Hilbert space as well), then that non-physical entity must not further be used in any further combinations in the hope to somehow retrieve its viability in physics. The function ψ_p is not viable physically from the get go.

If the above is not enough and one needs more proof for the non-physical and even mathematically non-sensical nature of $\psi_p(x)$, one may observe the dead-end it gets into when one tries to find the expectation value of the commutator of the position and momentum operators: $[\widehat{x}, \widehat{p}] = (\widehat{x}\widehat{p} - \widehat{p}\widehat{x}) = i\hbar$. What else can one expect when agreeing to use a non-physical entity as if it has physical meaning? The initial disaster will guarantee further equally as damaging disasters.

Thus, let us now see, just as an exercise, this disastrous dead-end in more detail. To find the expectation value of the commutator $[\widehat{x}, \widehat{p}]$ we must, in accordance with the expectation value postulate of quantum mechanics, sandwich it between either eigenfunction $\psi_p(x)$ or $\psi_x(x)$. We will choose the former and will express it, in the usual manner, by a ket-vector $|\psi_p(x)\rangle$:

$$\langle \psi_p(x) | (\widehat{x}\widehat{p} - \widehat{p}\widehat{x}) | \psi_p(x) \rangle = \langle \psi_p(x) | (-i\hbar) | \psi_p(x) \rangle \quad (8)$$

Now, let us observe separately the left side and the right side of eq.(8), simplifying it by writing just $|\psi_p\rangle$ instead of $|\psi_p(x)\rangle$

<u>left side of eq.(8)</u>

$$\langle \psi_p | (\widehat{x}\widehat{p} - \widehat{p}\widehat{x}) | \psi_p \rangle = \langle \psi_p | \widehat{x}\widehat{p} | \psi_p \rangle - \langle \psi_p | \widehat{p}\widehat{x} | \psi_p \rangle =$$

$$\langle \psi_p | \widehat{x} | \psi_p \rangle \, p - p \, \langle \psi_p | \widehat{x} | \psi_p \rangle =$$

$$\text{indeteminate}.p - p.\text{indeterminate} = \text{indeterminate} \quad (9)$$

because, $\langle \psi_p | \widehat{x} | \psi_p \rangle$ itself is indeterminate, which can be seen clearly when it is expressed in integral form:

$$\int_{-\infty}^{+\infty} \psi_p^* x \psi_p dx = \int_{-\infty}^{+\infty} Ce^{-i\frac{p}{\hbar}x} x Ce^{+i\frac{p}{\hbar}x} dx = \quad (10)$$

$$C^2 \int_{-\infty}^{+\infty} x dx = \text{indeterminate} \quad (11)$$

and <u>right side of eq.(8)</u>

$$\langle \psi_p | i\hbar | \psi_p \rangle = i\hbar \langle \psi_p | \psi_p \rangle = i\hbar.\infty = \infty \quad (12)$$

where, as already seen, $\langle\psi_p|\psi_p\rangle = \infty$, leading to

$$\text{indeterminate} = \infty \tag{13}$$

which is also a wrong equality brought about by quantum mechanics when abiding by its rules and not resorting to manipulating the obtained results.

"Smearing" of momentum ψ_p in an attempt to fudge a solution

The above problem is attempted to be "resolved" in an incredible way. Thus, instead of conceding that the above is a fatal problem and that quantum mechanics must be abandoned, quantum enthusiasts insist on a further use of the non-physical entity $\psi_p = Ce^{+i\frac{p}{\hbar}x}$, which, as shown, cannot serve as a state-function. What is done further in an attempt to make ψ_p appear to be in concordance with physical reality is to "smear" it (notice, how the real problem is sidestepped by observing the off-diagonal elements) and instead have[15]

$$\int_{-\infty}^{+\infty} \psi^*_{p_2}(x)\psi_{p_1}(x)dx = \int_{-\infty}^{+\infty} Ce^{-i\frac{p_2}{\hbar}x}Ce^{+i\frac{p_1}{\hbar}x}dx = \tag{14}$$

$$C^2\int_{-\infty}^{+\infty} e^{i\frac{p_1-p_2}{\hbar}x}dx = 2\pi\hbar C^2\delta(p_1-p_2), \tag{15}$$

In other words, the quantum enthusiasts are quick to claim that, if we write the eigenstates as $\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{i\frac{p}{\hbar}x}$ and apply the logical error *petitio principii*, we will have what these enthusiasts claim to be a properly normalized, in fact quasi-normalized, eigenfunction, its normalized value being equal to $\delta(p_1-p_2)$, which, it appears to them, is a non-zero value.

However, this is hardly a properly normalized eigenfunction. Firstly, it is obtained under the premise requiring without any grounds that **a single particle should have simultaneously two different values of momentum**. This is violation of logic (involving *petitio principii*).

Secondly, to add pain to suffering, even after resorting to a logical mistake (*petitio principii*) as an underhanded method to resolve the problem of normalization, the pseudo-normalized "function" $\delta(p_1-p_2)$ is also 0 because both p_1 and p_2 are constants, their difference is also constant and their δ-"function" is zero, as is any δ-"function" of a constant. Therefore, the pseudo-normalization did not do any good whatsoever.

Incredible as it may sound, a problem in quantum mechanics, which supposedly has been noticed by the founders[1], and which brings down the whole field, has been flied over throughout the decades, giving it an only occasional attention[2–5]. Additionally, it is a problem very easy to see let alone that it invalidates one of the most important features of quantum mechanics—the peculiar non-commutativity of operators—as well as the, probably, most important operators, those of position $\widehat{x}$ and momentum $\widehat{p}$. One may say, well, the above problem arises only when that particular product of ψ_p with its complex conjugate ψ^*_p is given a certain interpretation. However, if we consider what ψ_p, obtained by quantum mechanics truly is; namely, that it is a function not contained in the Hilbert space and is not square-integrable, then these problems lead to the Lebesgue integral $\langle\psi_p|\widehat{x}|\psi_p\rangle = \text{indeterminate}$. This very fact should preclude our interest in quantum mechanics altogether.

From the above, it turns out that the problems are deeper than just the ψ-function not being able to describe fully the system, as the critics of quantum mechanics have thought[10–12]. It turns out that the question is about the complete invalidity of said theory. Therefore, all the conclusions about quantum mechanics as being characterized by non-locality and entanglement leading to quantum computing or quantum teleportation are all non-physical. As seen, quantum mechanics is failing to derive its most important eigenfunctions—that of position and momentum.

This means also that the question for the interpretations of quantum mechanics, Copenhagen or otherwise, is moot. Also, because of the above, there cannot be any philosophical ramifications from quantum mechanics, as some tend to used to sound more learned and advanced.

Standard literature, however, most slyly skips these very important fatal points and leads the student to believe that somehow there is some strictly determined ψ_p existing ahead of time, from which the operator $\widehat{p}$ does not at all need to extract the concrete value p of the position because, the student is deceitfully led to believe, that said constant value p is also somehow known ahead of time. Once that wool is pulled over the eyes of the unsuspecting student, the quantum enthusiasts proceed to reveal to the wide-eyed disciple the incredibly fantastic world of quantum mechanics.

References and Notes

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13. In momentum space the position operator $\hat{x}$ is $i\hbar\frac{d}{dp}$ and the momentum operator $\hat{p}$ is p.
14. Just as an aside, notice that while the wave-function $\psi_p(x) = Ce^{+i\frac{p}{\hbar}x}$ is written in bra-ket notation simply as the ket-vector $|\psi_p\rangle = Ce^{+i\frac{p}{\hbar}x}$, the inner product $\langle\psi_p|\psi_p\rangle$, despite the natural supposition that $\langle\psi_p|\psi_p\rangle$ should be the simple inner product of ψ_p with its complex conjugate form ψ_p^*; that is, instead of writing it as $\langle\psi_p|\psi_p\rangle = \psi_p^*(x)\psi_p(x)$, is, instead, an integral; namely, the integral $\int\limits_{-\infty}^{+\infty}\psi_p^*(x)\psi_p(x)dx$. In other words, instead of just expressing the probability density, the bra-ket notation $\langle\psi_p|\psi_p\rangle$ already gives the probability.
15. Justification of the last equality; that is, that the integral $C^2\int_{-\infty}^{+\infty}e^{i(p_2-p_1)\frac{x}{\hbar}}dx$ is equivalent to the δ-function $2\pi\hbar C^2\delta(p_2-p_1)$ can be done easily from the Fourier transformations and the specification made by Couchy of the connection between the integral of an exponent and a δ-"function"